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WILLMORE SURFACES in S^n

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Abstract: A surface $x : M \to S^n$ is called a Willmore surface if it is a critical surface of the Willmore functional. It is well-known that any minimal surface is a Willmore surface and that there exist many non-minimal Willmore surfaces. In this paper, we establish an integral inequality for compact Willmore surfaces in S^n and get a new characterization of the Veronese surface in S^4 as a Willmore surface. Our result reduces to a well-known result in the case of minimal surfaces.

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1. Introduction

Let $x : M \to S^n$ be a surface in an n-dimensional unit sphere space S^n. If h^α_{ij} denotes the second fundamental form of M, S denotes the square of the length of the second fundamental form, $\mathbf{H}$ denotes the mean curvature vector and H denotes the mean curavture of M, then we have

$$S = \sum_\alpha \sum_{i,j} (h^\alpha_{ij})^2, \quad \mathbf{H} = \sum_\alpha H^\alpha e_\alpha, \quad H^\alpha = \frac{1}{2}\sum_k h^\alpha_{kk}, \quad H = |\mathbf{H}|,$$

where e_α $(3 \le \alpha \le n)$ are orthonormal vector fields of M in S^n.

We define the following non-negative function on M

$$\rho^2 = S - 2H^2, \tag{1.1}$$

which vanishes exactly at the umbilic points of M.

The Willmore functional is the following non-negative functional (see [2],[4] or [21])

$$W(x) = \int_M \rho^2 dv = \int_M (S - 2H^2) dv, \tag{1.2}$$

it was shown in [4] (also see [17], [19]) that this functional is an invariant under conformal transformations of S^n. The Willmore conjecture says that $W(x) \ge 4\pi^2$ holds for all immersed tori $x : M \to S^3$. The conjecture has been proved in some conformal classes by Li and Yau [12], Montiel and Ros [14]. The conjecture is also known to be true for

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flat tori (see Chen [5]) and tori whose images under stereographic projection are surfaces of revolution in R^3 (see Langer and Singer[10], Hertrich-Jeromin and Pinkall [8]). It is a natural idea to approach the Willmore conjecture by studying the critical surfaces of the Willmore functional $W(x)$. A surface in S^n is called a Willmore surface if it is a critical surface of the above Willmore functional.

Let M be a surface in S^n, it was proved by R. Bryant in case $n=3$ (see [2]) and by J. Weiner in [18] in the general case $n \geq 3$ that M is a Willmore surface if and only if

$$\Delta^{\perp} H^{\alpha} + \sum_{\beta,i,j} h_{ij}^{\alpha} h_{ij}^{\beta} H^{\beta} - 2H^2 H^{\alpha} = 0, \quad 3 \leq \alpha \leq 2+p. \tag{1.3}$$

Remark 1.1. From (1.3), it is obvious that all minimal surfaces in S^n are Willmore surfaces (see Weiner [18]). In [5], Pinkall constructed many compact non-minimal flat Willmore surfaces in S^3. In [3], Castro-Urbano constructed many compact non-minimal Willmore surfaces in S^4. In [7], Ejiri constructed a compact non-minimal flat Willmore surfaces in S^5.

In order to state our main result, we first recall the following example

Example 1 (see [6]). Veronese surface. Let (x, y, z) be the canonical coordinate system in R^3 and $u = (u_1, u_2, u_3, u_4, u_5)$ the canonical coordinate system in R^5. We consider the mapping defined by

$$u_1 = \frac{1}{\sqrt{3}} yz, \quad u_2 = \frac{1}{\sqrt{3}} xz, \quad u_3 = \frac{1}{\sqrt{3}} xy,$$

$$u_4 = \frac{1}{2\sqrt{3}}(x^2 - y^2), \quad u_5 = \frac{1}{6}(x^2 + y^2 - 2z^2),$$

where $x^2 + y^2 + z^2 = 3$. This defines an isometric immersion of $S^2(\sqrt{3})$ into $S^4(1)$. Two points (x, y, z) and $(-x, -y, -z)$ of $S^2(\sqrt{3})$ are mapped into the same point of S^4. This real projective plane imbedded in S^4 is called the *Veronese surface.* We know that the Veronese surface is a minimal surface in S^4 (see [6]), thus it is a Willmore surface. We also note that ρ^2 of the Veronese surface satisfies

$$\rho^2 = \frac{4}{3}. \tag{1.4}$$

In the theory of minimal sufaces in S^n, the following integral inequality is well-known

Theorem 1 (Benko-Kothe-Semmler-Simon [1] or Kozlowski-Simon [13]) *Let M be a compact minimal surface with Gauss curvature K in an n-dimensional unit sphere S^n. Then we have*

$$\int_M (1-K)(3K-1)dv \leq 0. \tag{1.5}$$

In particular, if

$$\frac{1}{3} \leq K \leq 1, \tag{1.6}$$

then either $K = 1$ and M is totally geodesic, or $K = \frac{1}{3}$, $n = 4$ and M is the Veronese surface given by Example 1.

For minimal surfaces in S^n, the Gauss equation reads $2K = 2 - S$, thus Theorem 1 is equivalent to

Theorem 2 *Let M be a compact minimal surface in an n-dimensional unit sphere S^n. Then we have*

$$\int_M S(2-\frac{3}{2}S)dv \leq 0. \tag{1.5}'$$

In particular, if

$$0 \leq S \leq \frac{4}{3}, \tag{1.6}'$$

then either $S=0$ and M is totally geodesic, or $S=\frac{4}{3}$, $n=4$ and M is the Veronese surface given by Example 1.

In this paper we prove the following integral inequality for compact Willmore surfaces in S^n.

Theorem 3 *Let M be a compact Willmore surface in an n-dimensional unit sphere S^n. Then we have*

$$\int_M \rho^2(2-\frac{3}{2}\rho^2)dv \leq 0. \tag{1.7}$$

In particular, if

$$0 \leq \rho^2 \leq \frac{4}{3}, \tag{1.8}$$

then either $\rho^2=0$ and M is totally umbilic, or $\rho^2=\frac{4}{3}$. In the latter case, $n=4$ and M is the Veronese surface given by example 1.

Remark 1.2. In the case of minimal surfaces, Theorem 3 reduces to Theorem 2.

Remark 1.3. The author [11] proved the following result for compact Willmore surfaces in S^3

Theorem 4 (see Theorem 4 of [11]) *Let M be a compact Willmore surface in S^3. Then we have*

$$\int_M \rho^2(2-\rho^2)dv \leq 0. \tag{1.9}$$

In particular, if

$$0 \leq \rho^2 \leq 2, \tag{1.10}$$

then either $\rho^2=0$ and M is totally umbilic, or $\rho^2=2$ and

$$M = S^1(\sqrt{\frac{1}{2}}) \times S^1(\sqrt{\frac{1}{2}}).$$

2. Preliminaries

Let $x: M \to S^n$ be a surface in an n-dimensional unit sphere. We choose an orthonormal basis $e_1, \cdots, e_n$ of S^n such that $\{e_1, e_2\}$ are tangent to $x(M)$ and $\{e_3, \cdots, e_n\}$ is a local frame in the normal bundle. Let $\{\omega_1, \omega_2\}$ be the dual forms of $\{e_1, e_2\}$. We use the following convention on the ranges of indices:

$$1 \leq i, j, k, \cdots \leq 2; \qquad 3 \leq \alpha, \beta, \gamma, \cdots \leq n.$$

Then we have the structure equations

$$dx = \sum_i \omega_i e_i, \tag{2.1}$$

$$de_i = \sum_j \omega_{ij} e_j + \sum_{\alpha,j} h^\alpha_{ij}\omega_j e_\alpha - \omega_i x, \tag{2.2}$$

$$de_\alpha = -\sum_{i,j} h^\alpha_{ij}\omega_j e_i + \sum_\beta \omega_{\alpha\beta} e_\beta, \quad h^\alpha_{ij} = h^\alpha_{ji}. \tag{2.3}$$

The Gauss equations and Ricci equations are

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}), \tag{2.4}$$

$$R_{ik} = \delta_{ik} + 2\sum_\alpha H^\alpha h^\alpha_{ik} - \sum_{\alpha,j} h^\alpha_{ij}h^\alpha_{jk}, \tag{2.5}$$

$$2K = 2 + 4H^2 - S, \tag{2.6}$$

$$R_{\beta\alpha 12} = \sum_i (h^\beta_{1i}h^\alpha_{i2} - h^\beta_{2i}h^\alpha_{i1}), \tag{2.7}$$

where K is the Gauss curvature of M and $S = \sum_{\alpha,i,j}(h^\alpha_{ij})^2$ is the square of the norm of the second fundamental form; $\mathbf{H} = \sum_\alpha H^\alpha e_\alpha = \frac{1}{2}\sum\limits_\alpha(\sum\limits_k h^\alpha_{kk})e_\alpha$ is the mean curvature vector and $H = |\mathbf{H}|$ is the mean curvature of M.

By Gauss equation (2.6), (1.2) becomes

$$W(x) = 2\int_M (H^2 - K + 1)dv. \tag{2.8}$$

It was shown in [4] and [19] that this functional is an invariant under conformal transformations of S^n.

We have the following Codazzi equations and Ricci identities

$$h^\alpha_{ijk} - h^\alpha_{ikj} = 0, \tag{2.9}$$

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{mj}R_{mikl} + \sum_m h^\alpha_{im}R_{mjkl} + \sum_\beta h^\beta_{ij}R_{\beta\alpha kl}, \tag{2.10}$$

where h^α_{ijk} and h^α_{ijkl} are defined by

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} + \sum_k h^\alpha_{kj}\omega_{ki} + \sum_k h^\alpha_{ik}\omega_{kj} + \sum_\beta h^\beta_{ij}\omega_{\beta\alpha} \tag{2.11}$$

$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} + \sum_l h^\alpha_{ljk}\omega_{li} + \sum_l h^\alpha_{ilk}\omega_{lj} + \sum_l h^\alpha_{ijl}\omega_{lk} + \sum_\beta h^\beta_{ijk}\omega_{\beta\alpha}. \tag{2.12}$$

As M is a 2-dimensional surface, we have from (2.6) and (1.1)

$$2K = 2 + 4H^2 - S = 2 + 2H^2 - \rho^2, \tag{2.13}$$

$$R_{ijkl} = K(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \quad R_{ik} = K\delta_{ik}. \tag{2.14}$$

Lemma 2.1 *Let $x : M \to S^n$ be a surface. Then we can write, for $3 \le \alpha \le n$,*

$$\frac{h^\alpha_{11} - h^\alpha_{22}}{2} = \lambda\cos\theta_\alpha, \quad \sum_\alpha \cos^2\theta_\alpha = 1, \quad \lambda \ge 0, \tag{2.15}$$

$$h_{12}^{\alpha} = \mu\cos\phi_{\alpha}, \quad \sum_{\alpha}\cos^2\phi_{\alpha} = 1, \quad \mu \geq 0, \tag{2.16}$$

$$\rho^2 = 2(\lambda^2 + \mu^2). \tag{2.17}$$

Proof. By choosing

$$\lambda = \frac{1}{2}\sqrt{\sum_{\alpha}(h_{11}^{\alpha} - h_{22}^{\alpha})^2}, \quad \mu = \sqrt{\sum_{\alpha}(h_{12}^{\alpha})^2}, \tag{2.18}$$

we get (2.15) and (2.16). (2.17) comes from (1.1) and (2.18).

Lemma 2.2 *Let $x : M \to S^n$ be a surface. Then we have*

$$\begin{aligned} &\tfrac{1}{2}\Delta \sum_{\alpha,i,j}(h_{ij}^{\alpha})^2 \\ \geq\ & |\nabla h|^2 + \sum_{\alpha,i,j,k}(h_{ij}^{\alpha}h_{kki}^{\alpha})_j - 4|\nabla^{\perp}\mathbf{H}|^2 + 2(1+H^2)\rho^2 - \tfrac{3}{2}\rho^4, \end{aligned} \tag{2.19}$$

where $|\nabla h|^2 = \sum(h_{ijk}^{\alpha})^2$ and $|\nabla^{\perp}\mathbf{H}|^2 = \sum(H_i^{\alpha})^2$.

Proof. Using (2.7), (2.9), (2.10) , (2.13) and (2.14), we have the following calculations

$$\begin{aligned} &\tfrac{1}{2}\Delta \sum_{\alpha,i,j}(h_{ij}^{\alpha})^2 \\ =\ & \sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2 + \sum_{\alpha,i,j,k} h_{ij}^{\alpha}h_{ijkk}^{\alpha} \\ =\ & \sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2 + \sum_{\alpha,i,j,k} h_{ij}^{\alpha}h_{kkij}^{\alpha} + \sum(h_{ij}^{\alpha}h_{km}^{\alpha}R_{mijk} + h_{ij}^{\alpha}h_{mi}^{\alpha}R_{mj}) \\ & + \sum h_{ij}^{\alpha}h_{ki}^{\beta}R_{\beta\alpha jk} \\ =\ & \sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2 + \sum_{\alpha,i,j} h_{ij}^{\alpha}h_{kkij}^{\alpha} + 2K\rho^2 + \sum_{\alpha,\beta}[\sum_{i}(h_{1i}^{\alpha}h_{i2}^{\beta} - h_{2i}^{\alpha}h_{i1}^{\beta})]R_{\beta\alpha 12} \\ =\ & \sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2 + \sum_{\alpha,i,j,k} h_{ij}^{\alpha}h_{kkij}^{\alpha} + 2K\rho^2 - \sum_{\alpha,\beta}(R_{\beta\alpha 12})^2. \end{aligned} \tag{2.20}$$

From (2.7), (2.15) and (2.16)

$$\begin{aligned} R_{\beta\alpha 12} &= \sum_{i=1}^{2}(h_{1i}^{\beta}h_{i2}^{\alpha} - h_{2i}^{\beta}h_{i1}^{\alpha}) \\ &= (h_{11}^{\beta} - h_{22}^{\beta})h_{12}^{\alpha} - (h_{11}^{\alpha} - h_{22}^{\alpha})h_{12}^{\beta} \\ &= 2\lambda\mu(\cos\theta_{\beta}\cos\phi_{\alpha} - \cos\theta_{\alpha}\cos\phi_{\beta}). \end{aligned} \tag{2.21}$$

Putting (2.21) into (2.20) and using (2.13), we have

$$\begin{aligned} &\tfrac{1}{2}\Delta \sum_{\alpha,i,j}(h_{ij}^{\alpha})^2 \\ =\ & \sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2 + \sum_{\alpha,i,j} h_{ij}^{\alpha}h_{kkij}^{\alpha} + 2K\rho^2 - 4\lambda^2\mu^2\sum_{\alpha,\beta}(\cos\theta_{\beta}\cos\phi_{\alpha} - \cos\theta_{\alpha}\cos\phi_{\beta})^2 \\ =\ & \sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2 + \sum_{\alpha,i,j} h_{ij}^{\alpha}h_{kkij}^{\alpha} + 2K\rho^2 \\ & -4\lambda^2\mu^2[2\sum_{\alpha,\beta}\cos^2\theta_{\beta}\cos^2\phi_{\alpha} - 2\sum_{\alpha,\beta}\cos\theta_{\beta}\cos\theta_{\alpha}\cos\phi_{\beta}\cos\phi_{\alpha}] \\ =\ & |\nabla h|^2 + \sum_{\alpha,i,j,k}(h_{ij}^{\alpha}h_{kki}^{\alpha})_j - \sum_{\alpha,i,j,k} h_{ijj}^{\alpha}h_{kki}^{\alpha} + 2K\rho^2 - 4\lambda^2\mu^2[2 - 2(\sum_{\alpha}\cos\theta_{\alpha}\cos\phi_{\alpha})^2] \\ \geq\ & |\nabla h|^2 + \sum_{\alpha,i,j,k}(h_{ij}^{\alpha}h_{kki}^{\alpha})_j - 4\sum_{\alpha,i}(H_i^{\alpha})^2 + 2(1+H^2)\rho^2 - \tfrac{3}{2}\rho^4, \end{aligned} \tag{2.22}$$

for the inequality we used

$$8\lambda^2\mu^2 \le 2(\lambda^2+\mu^2)^2 = \frac{1}{2}\rho^4.$$

The following Euler-Lagrange equation for the Willmore functional was derived by J. Weiner in [18].

Lemma 2.3 . *Let $x : M \to S^n$ be a surface in an n-dimensional unit sphere S^n. Then M is a Willmore surface if and only if*

$$\Delta^{\perp}H^{\alpha} + \sum_{\beta,i,j} h_{ij}^{\alpha}h_{ij}^{\beta}H^{\beta} - 2H^2H^{\alpha} = 0, \quad 3 \le \alpha \le n. \tag{2.23}$$

We also need the following lemma to prove our Theorem 3

Lemma 2.4 *Let M be a surface in S^n, then we have*

$$|\nabla h|^2 \ge 3|\nabla^{\perp}\mathbf{H}|^2, \tag{2.24}$$

where $|\nabla h|^2 = \sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2$, $|\nabla^{\perp}\mathbf{H}|^2 = \sum_{\alpha,i}(H_i^{\alpha})^2$, $H_i^{\alpha} = \nabla_i H^{\alpha}$.

Proof. We construct the following symmetric tracefree tensor

$$F_{ijk}^{\alpha} = h_{ijk}^{\alpha} - \frac{1}{2}(H_i^{\alpha}\delta_{jk} + H_j^{\alpha}\delta_{ik} + H_k^{\alpha}\delta_{ij}). \tag{2.25}$$

Then we can easily compute that

$$|F|^2 = \sum_{\alpha,i,j,k}(F_{ijk}^{\alpha})^2 = |\nabla h|^2 - 3|\nabla^{\perp}\mathbf{H}|^2.$$

We get

$$|\nabla h|^2 \ge 3|\nabla^{\perp}\mathbf{H}|^2,$$

which proves Lemma 2.4.

Remark 2.1. The analogue of Lemma 2.4 for hypersurfaces in S^n can be found in [1], [9] and [11].

Now we define the following tracefree tensor

$$\tilde{h}_{ij}^{\alpha} = h_{ij}^{\alpha} - H^{\alpha}\delta_{ij}, \tag{2.26}$$

then Lemma 2.3 becomes

Lemma 2.5 . *Let $x : M \to S^n$ be a surface in an n-dimensional unit sphere S^n. Then M is a Willmore surface if and only if*

$$\Delta^{\perp}H^{\alpha} + \sum_{\beta,i,j} \tilde{h}_{ij}^{\alpha}\tilde{h}_{ij}^{\beta}H^{\beta} = 0, \quad 3 \le \alpha \le n. \tag{2.27}$$

Lemma 2.6 *Let $x : M \to S^n$ be a Willmore surface then*

$$\int_M |\nabla^{\perp}\mathbf{H}|^2 = \int_M \sum_{\alpha,i}(H_i^{\alpha})^2 = \int_M \sum_{\alpha,\beta}\tilde{\sigma}_{\alpha\beta}H^{\alpha}H^{\beta}, \tag{2.28}$$

where

$$\tilde{\sigma}_{\alpha\beta} = \sum_{i,j} \tilde{h}^{\alpha}_{ij}\tilde{h}^{\beta}_{ij}. \tag{2.29}$$

Proof. By use of (2.27), we have

$$\begin{aligned}
|\nabla^{\perp}\mathbf{H}|^2 &= \sum_{\alpha,i}(H^{\alpha}_i)^2 \\
&= \sum_{\alpha,i}(H^{\alpha}H^{\alpha}_i)_i - \sum_{\alpha} H^{\alpha}\Delta^{\perp}H^{\alpha} \\
&= \sum_{\alpha,i}(H^{\alpha}H^{\alpha}_i)_i + \sum_{\alpha,\beta} \tilde{\sigma}_{\alpha\beta}H^{\alpha}H^{\beta}.
\end{aligned} \tag{2.30}$$

We get (2.28) by integrating (2.30) over M.

We note that the $(n-2)\times(n-2)$-matrix $(\tilde{\sigma}_{\alpha\beta})$ is symmetric, then it can be assumed to be diagonal for a suitable choice of $e_3, \cdots, e_n$. We set

$$\tilde{\sigma}_{\alpha\beta} = \tilde{\sigma}_{\alpha}\delta_{\alpha\beta}. \tag{2.31}$$

By use of (2.8), (2.26) and (2.29), we have

$$\rho^2 = \sum_{\alpha} \tilde{\sigma}_{\alpha}. \tag{2.32}$$

3. Proof of Theorem 3

In this section, we give the proof of Theorem 3.

Integrating (2.19) over M, by use of Lemma 2.4 and Lemma 2.6 we have

$$\begin{aligned}
0 &\geq \int_M[(|\nabla h|^2 - 3|\nabla^{\perp}\mathbf{H}|^2) - |\nabla^{\perp}\mathbf{H}|^2 + 2(1+H^2)\rho^2 - \tfrac{3}{2}\rho^4] \\
&\geq \int_M[-|\nabla^{\perp}\mathbf{H}|^2 + 2(1+H^2)\rho^2 - \tfrac{3}{2}\rho^4] \\
&= \int_M(2-\tfrac{3}{2}\rho^2)\rho^2 + \int_m H^2\rho^2 + \int_M(H^2\rho^2 - \sum_{\alpha,\beta} H^{\alpha}H^{\beta}\tilde{\sigma}_{\alpha\beta}) \\
&\geq \int_M(2-\tfrac{3}{2}\rho^2)\rho^2 + \int_M(H^2\rho^2 - \sum_{\alpha,\beta} H^{\alpha}H^{\beta}\tilde{\sigma}_{\alpha\beta}).
\end{aligned} \tag{3.1}$$

From (2.31) and (2.32), we get

$$H^2\rho^2 = (\sum_{\alpha}(H^{\alpha})^2)(\sum_{\beta}\tilde{\sigma}_{\beta}) \geq \sum_{\alpha}(H^{\alpha})^2\tilde{\sigma}_{\alpha} = \sum_{\alpha,\beta} H^{\alpha}H^{\beta}\tilde{\sigma}_{\alpha\beta}. \tag{3.2}$$

Putting (3.2) into (3.1), we reach the following integral inequality

$$\int_M \rho^2(2-\frac{3}{2}\rho^2) \leq 0. \tag{3.3}$$

Therefore we have proved the integral inequality (1.11) in Theorem 3.

If (1.12) holds, then from (3.3) we conclude that either $\rho^2 \equiv 0$, or $\rho^2 \equiv \frac{4}{3}$. In the first case, we know that $S \equiv 2H^2$, i.e. M is totally umbilic; in the latter case, i.e., $\rho^2 \equiv \frac{4}{3}$, (3.1) becomes an equality, thus we have

$$\int_M H^2\rho^2 = 0. \tag{3.4}$$

(3.4) implies that $H = 0$, thus $x : M \to S^n$ is a minimal surface with

$$S = \frac{4}{3}.$$

From Theorem 2 we can conclude that $n = 4$ and $x : M \to S^4$ is a Veronese surface, which is given by Example 1. We complete the proof of Theroem 3.

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